$$(A_1+A_5)(A_3-A_2)=(-A_1-A_3+A_2+A_4-A_3+A_6+A_2-A_5)$$
$$=-\{1+(A_3-A_2)\}.$$

Similarly, $(A_1+A_5)(-A_4-A_6)=-\{1+(A_1+A_5)\}$; $(A_3-A_2)(-A_4-A_6)=-\{1+(-A_4-A_6)\}$.

$\therefore\ \{(A_1+A_5)(A_3-A_2)+(A_1+A_5)(-A_4-A_6)+(A_3-A_2)(-A_4-A_6)\}=-4.$

Now $(A_1+A_5)(A_3-A_2)(-A_4-A_6)=-(-A_4-A_6)-(A_3-A_2)(-A_4-A_6=A_4+A_6+1+(-A_4-A_6)=1.$

Hence, $(A_1+A_5),(A_3-A_2),(-A_4-A_6)$ are the three roots of the cubic $x^3-x^2-4x-1=0$. Call them A, B, C, respectively.

$$\therefore\ A_1+A_5=A;\ A_1.A_5=(A_4+A_6)=-C.$$
$$A_3-A_2=B;\ -A_3.A_2=-(A_1+A_5)=-A.$$
$$-A_4-A_6=C:\ A_4.A_6=(A_2-A_3)=-B.$$

Hence, A_1 and A_5 are the roots of $x^2-Ax-C=0$.

A_3 and $-A_2$ are the roots of $x^2-Bx-A=0$.

$-A_4$ and $-A_6$ are the roots of $x^2-Cx-B=0$.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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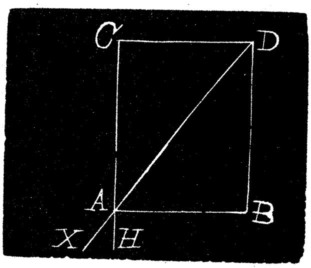
[Continued from the September Number.]

Given any triangle (fig.7) ABD right angled at B; prolong DA at any point X, and through A erect HAC perpendicular to AB.



I say the external angle XAH will be equal, or less, or greater than the internal and opposite ADB, according as is true the hypothesis of right angle, or obtuse angle, or acute angle: and inversely.

Proof. Assume in HC the portion AC equal to BD, and join CD. CD will be, in the hypothesis of right angle (P. III.) equal to AB. Wherefore the angle ADB will be equal (Eu. I. 8.) to the angle DAC, or to its equal (Eu. I. 15.) to the angle XAH. Quod erat primo loco demonstrandum.

Then, in the hypothesis of obtuse angle, CD will be (P. III.) less than AB.

Wherefore in the triangle ADB, the angle DAC, or its vertical XAH, will be (Eu. I. 25.) less than the angle ADB. Quod erat secundo loco demonstrandum.

While; in the hypothesis of acute angle, CD will be (P. III.) greater than the opposite AB. Wherefore in the said triangle the angle DAC, or its verticle XAH, will be (Eu. I. 25.) greater than the angle ADA. Quod erat tertio loco demonstrandum.

But inversely: if the angle CAD, or its vertical XAH, be equal to the internal and opposite ADB; the join CD will be (Eu. I. 4.) equal to AB, and therefore the hypothesis of right angle will be (P. IV.) true.

But if however the angle CAD,or its vertical XAH, be less, or greater than the internal or opposite ADB; also the join CD will be (Eu. I. 24.) less or greater than AB; and therefore (P. IV.) will be true respectively the hypothesis of obtuse angle, or acute angle. Quod omnia erant demonstranda.

Proposition IX. In any right-angled triangle the two acute angles remaining are taken together equal to one right angle, in the hypothesis of right angle; greater than one right angle, in the hypothesis of obtuse angle; but less in the hypothesis of acute angle.

Proof. For if the angle XAH (fig. 7.) is equal to the angle ADB, which is certain, from the preceding proposition, in the hypothesis of right angle, then the angle ADB makes up with the angle HAD two right angles, as (Eu. I. 13.) the aforesaid angle XAH makes them up with this angle HAB being subtracted, the two angles ADB and BAD remain together equal to one right angle. Quod erat primum.

However, if the angle XAH is less than the angle ADB, which is certain from the preceding proposition, in the hypothesis of obtuse angle, then the angle ADB makes up with the angle HAD more than two right angles, since with it (Eu. I. 13.) the angle XAH makes up two. Wherefore, the angle HAB being subtracted, the two angles ADB and BAD will be together greater than one right angle. Quod erat secundum.

Finally, if the angle XAH be greater than the angle ADB, which is certain from the preceding proposition in the hypothesis of acute angle, then the angle ADB will make up less than two right angles with the angle HAD, since with this (Eu. I. 13.) the angle XAH makes up two. Wherefore, subtracting the right angle HAB, the angles ADB and BAD will be together less than one right angle. Quod erat tertium.